Dylan Yott

Maximal Varieties and Representation Theory

Honors Thesis under the Supervision of Jared Weinstein

1 Introduction

Let X be a geometrically connected nonsingular projective curve of genus g over $\mathbb{F}_q$. Then by a result of Weil and Deligne, we have that the number of points on X over $\mathbb{F}_{q^n}$ satisfies the following bound:

$$\#X(\mathbb{F}_{q^n}) \leq q^n + 1 + 2g\sqrt{q^n}$$

This formula has an interesting cohomological interpretation as follows. Fix a prime $\ell \not| q$ for the remainder of the paper. Consider the Frobenius endomorphism, $Frob_q : X \to X$. The Frobenius endomorphism also induces endomorphisms on the étale cohomology of X, $H^i_{ét}(X, \overline{\mathbb{Q}_\ell})$. We can identify $X(\mathbb{F}_{q^n})$ with the fixed points of $Frob_q^n$, and then apply the Grothendieck-Lefschetz fixed point theorem to obtain the following:

$$\begin{aligned}\#X(\mathbb{F}_{q^n}) &= \sum_{i=0}^{2}(-1)^i Tr((Frob_q)^n | H^i_{ét}(X, \overline{\mathbb{Q}_\ell})) \\ &= 1 - \sum_{k=1}^{2g} \beta_k + q^n\end{aligned}$$

Here, β_k is the k^{th} eigenvalue of $(Frob_q)^n$ acting on $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})$. We've used a result of Deligne that $Frob_q$ acts on $H^0_{ét}(X, \overline{\mathbb{Q}_\ell})$ and $H^2_{ét}(X, \overline{\mathbb{Q}_\ell})$ by the scalars 1 and q respectively. Intuitively, we should expect these to be true, as $H^0_{ét}(X, \overline{\mathbb{Q}_\ell})$ is one-dimensional and keeps track of connected components, while $H^2_{ét}(X, \overline{\mathbb{Q}_\ell})$ is also one-dimensional and detects the degree of $Frob_q$, which is q. More generally, Deligne proved that the modulus of the eigenvalues of $Frob_q$ on $H^i_{ét}(X, \overline{\mathbb{Q}_\ell})$ are bounded by $q^{\frac{i}{2}}$, which allows us to bound $\#X(\mathbb{F}_{q^n})$ as follows:

$$\begin{aligned}\#X(\mathbb{F}_{q^n}) &\leq 1 - dim(H^1_{ét}(X, \overline{\mathbb{Q}_\ell}))(-\sqrt{q}) + q^n \\ &= 1 + q^n + 2g\sqrt{q^n}\end{aligned}$$

We are interested in curves that have the maximal number of points with respect to this bound, which we make clear in the following definition.

Definition 1.1. *Let X be a geometrically connected nonsingular projective curve of genus g defined over $\mathbb{F}_q$. We say X is maximal over $\mathbb{F}_{q^n}$ if*

$$\#X(\mathbb{F}_{q^n}) = 1 + q^n + 2g\sqrt{q^n}$$

It is an important remark that X is maximal over $\mathbb{F}_{q^n}$ iff $Frob_q^n$ acts on $H^i_{ét}(X, \overline{\mathbb{Q}_\ell})$ by the scalar $(-1)^i q^{\frac{ni}{2}}$.

We have a few examples of maximal varieties:

Consider the projective curve $H : y^q z + y z^q = x^{q+1}$ over $\mathbb{F}_{q^2}$. This has one point at infinity, $[1,0,0]$. To count the rest of the points, we consider the affine part of H given by $y^q + y = x^{q+1}$. Conveniently, this is the same as $Tr(y) = Nm(x)$, where Tr and Nm are the trace and norm maps from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. Fix an $x \in \mathbb{F}_{q^2}$. Then since the trace is a surjective linear map, we see that there are q values of y with $Tr(y) = Nm(x)$. This gives

$$q^2 \cdot (q+1) = q^3 + 1$$

points on $H(\mathbb{F}_{q^2})$. By the degree-genus formula, we see that the genus of X, g is given by:

$$g = \frac{(d-1)(d-2)}{2} = \frac{q(q-1)}{2}$$

Putting this together yields:

$$\begin{aligned} q^2 + 1 + 2g\sqrt{q^2} &= q^2 + 1 + q^3 - q^2 \\ &= q^3 + 1 \end{aligned}$$

Thus, H is maximal over $\mathbb{F}_{q^2}$.

In the elliptic curve case ($g = 1$), supersingular curves give an example of maximal curves in the following way. Let $p > 3$ and suppose $E/\mathbb{F}_p$ is supersingular. Then it is a standard result that this is true iff $\#E(\mathbb{F}_p) = p + 1$. By the Weil conjectures, the eigenvalues of $Frob_p$ on $H^1_{\acute{e}t}(E, \overline{\mathbb{Q}_\ell})$ are $\pm i\sqrt{p}$. Then,

$$\begin{aligned} \#E(\mathbb{F}_{p^2}) &= p^2 + 1 - (i\sqrt{p})^2 - (-i\sqrt{p})^2 \\ &= p^2 + 1 + 2p \end{aligned}$$

which tells us that E is maximal over $\mathbb{F}_{p^2}$.

In these examples, we've used the trace formula to put an upper bound on the number of points a variety has, however we can also use this formula to put a lower bound on the number of points. In particular, these geometric techniques can prove that certain varieties have any points at all. Here is an interesting example.

Consider the Fermat quartic:

$$F : x^4 + y^4 + z^4 + w^4 = 0$$

It turns out that F considered over $\mathbb{F}_q$ is the reduction of a K3 surface S whose middle cohomology is 22-dimensional. Thus we have the following bound:

$$\#F(\mathbb{F}_q) \geq q^2 - 22q + 1$$

Thus, F has a point over $\mathbb{F}_q$ whenever $q > 22$. By Fermat's little theorem, F has no points over $\mathbb{F}_5$. It is easy to find points for $q = 2, 3, 7, 8, 9, 11, 13, 16, 17$, and 19. Thus,

F has points over every finite field except $\mathbb{F}_5$.

Our study of maximal varieties will be intimately related with finite group actions on varieties over finite fields, and the resulting representations coming from the cohomology. In the following sections, we work through an extended example in which we explicitly produce a certain class of representations of $GL_2(\mathbb{F}_q)$ in the cohomology of a certain variety.

2 Representation Theory

For $\alpha \in \mathbb{F}_q^\times$, consider the curves $X_\alpha : xy^q - x^q y = \alpha z^{q+1}$. Consider their disjoint union as a projective variety $X \in \mathbb{P}^2(\overline{\mathbb{F}_q})$. Since each X_α is connected, we see that the connected components of X are in bijection with $\mathbb{F}_q^\times$. It turns out that X has an action of $GL_2(\mathbb{F}_q)$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x, y, z] = [ax + by, cx + dy, z]$$

A matrix $A \in GL_2(\mathbb{F}_q)$ sends a point in X_α to a point in $X_{det(A)\alpha}$. We also have an action of $\mathbb{F}_{q^2}^\times$ on X as follows. For $\beta \in \mathbb{F}_{q^2}^\times$:

$$\beta \cdot [x, y, z] = [\beta x, \beta y, z]$$

This sents a point in X_α to a point in $X_{Nm(\beta)\alpha}$. A crucial fact to notice here is that the actions of $GL_2(\mathbb{F}_q)$ and $\mathbb{F}_{q^2}^\times$ commute, as:

$$\begin{aligned} A \cdot (\beta \cdot [x, y, z]) &= A \cdot [\beta x, \beta y, z] \\ &= [\beta ax + \beta by, \beta cx + \beta dy] \\ &= \beta \cdot (A \cdot [x, y, z]) \end{aligned}$$

This will be important, as it means the actions these groups induce on $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})$ will commute, and will allow us better understand the action of $GL_2(\mathbb{F}_q)$ using the action of the simpler (abelian, in fact cyclic) group $\mathbb{F}_{q^2}^\times$. For notational purposes, we will refer to the representations of $\mathbb{F}_{q^2}^\times$ and $GL_2(\mathbb{F}_q)$ on $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})$ as ρ_C and ρ_M respectively. The goal of this example is to understand ρ_C and leverage this information to show that ρ_M breaks up into a special class of representations called cuspidal representations. First, we need the following definition.

Definition 2.1. *A character $\chi : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{Q}_\ell}^\times$ is called generic if it doesn't factor through the norm map to $\mathbb{F}_q^\times$. Equivalently for $\alpha \in \mathbb{F}_{q^2}^\times$ a generator, $\chi(\alpha)$ is not a $(q-1)^{st}$ root of unity.*

Of the $(q^2 - 1)$ characters of $\mathbb{F}_{q^2}^\times$, $q^2 - q$ of them are generic. With this definition in place, we can state the following proposition:

Proposition 2.2. $\rho_C = \bigoplus_{\chi \text{ generic}} \chi^{\oplus(q-1)}$.

Proof. We begin the proof by computing $Tr(\rho_C(\alpha))$ for $\alpha \in \mathbb{F}_{q^2}^\times$. As usual, we use the Lefschetz trace formula:

$$Tr(\rho_C(\alpha)) = Tr([\alpha])|H^0_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell}) + Tr([\alpha])|H^2_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell}) - \#Fix([\alpha])$$

We tackle the computation in cases.

$\boxed{\alpha = 1}$

In this case:

$$\begin{aligned} Tr(\rho_C(1)) &= dim(H^1_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell})) \\ &= (q-1)dim(H^1_{\acute{e}t}(X_\alpha, \overline{\mathbb{Q}_\ell})) \\ &= q(q-1)^2 \end{aligned}$$

The last equality equality comes from the fact that each X_α has genus $\frac{q(q-1)}{2}$.

$\boxed{Nm(\alpha) \neq 1}$

If $Nm(\alpha) \neq 1$, then $Tr(\rho_C(\alpha)) = 0$, as that means α doesn't even fix connected components, so it certainly doesn't fix any points. Since $[\alpha]$ is a degree 1 automorphism permuting the connected components, and $H^0_{\acute{e}t}(X_\alpha, \overline{\mathbb{Q}_\ell})$ and $H^2_{\acute{e}t}(X_\alpha, \overline{\mathbb{Q}_\ell})$ are one-dimensional, we see that $[\alpha]|H^0_{\acute{e}t}(X_\alpha, \overline{\mathbb{Q}_\ell})$ and $[\alpha]|H^2_{\acute{e}t}(X_\alpha, \overline{\mathbb{Q}_\ell})$ are just $(q-1)$-dimensional permutation matrices. When $Nm(\alpha) \neq 1$ this permutation doesn't fix any connected components, and thus has trace 0.

$\boxed{Nm(\alpha) = 1, \alpha \neq 1}$

In this case for $i = 0, 2$, $[\alpha]|H^i_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell})$ is the identity so

$$Tr(\alpha)|H^i_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell}) = q - 1$$

Now we need to compute the number of fixed points, so suppose $[x, y, z] = [\alpha x, \alpha y, z]$. We must have $z = 0$, otherwise this implies $x = \alpha x$. The points with $z = 0$ are called the "points at infinity", and they are all clearly fixed, as $[x, y, 0] = [\alpha x, \alpha y, 0]$. The number of points at infinity in each X_α is $q + 1$ and they are naturally in bijection with $\mathbb{P}^1(\mathbb{F}_q)$. These facts together give:

$$\begin{aligned} Tr(\rho_C(\alpha)) &= (q-1) + (q-1) - (q+1)(q-1) \\ &= -(q-1)^2 \end{aligned}$$

Now that we've computed all of the traces, we compute the multiplicity of each generic χ in ρ_C and show that it is $(q-1)$. Since there are $q(q-1)$ such generic characters, and $dim(H^1_{ét}(X, \overline{\mathbb{Q}_\ell})) = q(q-1)^2$, we will have completely decomposed ρ_C. We compute the following inner product for χ generic:

$$\rho_C \cdot \chi = \frac{1}{q^2-1} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} Tr(\rho_C(\alpha))\chi(\alpha) \tag{1}$$

$$= \frac{1}{q^2-1} \left(q(q-1)^2 + \sum_{Nm(\alpha)=1, \alpha \neq 1} -(q-1)^2 \chi(\alpha) \right) \tag{2}$$

$$= \frac{1}{q^2-1} \left(q(q-1)^2 + (q-1)^2 \right) \tag{3}$$

$$= q-1 \tag{4}$$

The genericity of χ implies that $\{\chi(\alpha) | Nm(\alpha) = 1\}$ doesn't only contain 1, forcing the inner sum in (2) to collapse to -1 by a traditional character sums argument. Although we don't need to show it since we've already matched up the dimensions, we can see that ρ_C doesn't contain any non-generic χ directly by observing that $\chi(\alpha) = \chi(Nm(\alpha)) = 1$ for α with norm 1. Thus the sum would collapse completely in that case and the multiplicity would be 0. Thus we have the desired decomposition of ρ_C. □

3 A Langlands Correspondence

Our work with ρ_C will allow us to determine the decomposition of ρ_M into irreducible representations and state a correspondence similar to the Langlands correspondence. For $\chi : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{Q}_\ell}^\times$, we can consider $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})[\chi]$, the χ-isotypic component of ρ_C, which is defined as follows:

$$H^1_{ét}(X, \overline{\mathbb{Q}_\ell})[\chi] = \{v \in H^1_{ét}(X, \overline{\mathbb{Q}_\ell}) |\ \rho_C(\alpha)(v) = \chi(\alpha)v, \forall \alpha \in \mathbb{F}_{q^2}^\times\}$$

Clearly $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})[\chi]$ is a representation of $\mathbb{F}_{q^2}^\times$, and in fact, by the calculation in the previous section, this will be $(q-1)$-dimensional when χ is generic, and trivial otherwise. Since the actions of $\mathbb{F}_{q^2}^\times$ and $GL_2(\mathbb{F}_q)$ commute, we actually have that $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})[\chi]$ is a representation of $GL_2(\mathbb{F}_q)$, and we even have a formula for the trace of an element $A \in GL_2(\mathbb{F}_q)$ on $H^1_{ét}(X, \overline{\mathbb{Q}_\ell})[\chi]$ as follows:

$$Tr(A)|H^1_{ét}(X, \overline{\mathbb{Q}_\ell})[\chi] = \frac{1}{|\mathbb{F}_{q^2}^\times|} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \chi(\alpha) Tr(\alpha^{-1} \circ A) | H^1_{ét}(X, \overline{\mathbb{Q}_\ell})$$

This formula follows in a straightforward way from the orthogonality relations. To understand ρ_M, we need only to understand its restrictions to the $\rho_C[\chi]$, and using some facts about the representation theory of $GL_2(\mathbb{F}_q)$, we can show that this restriction is either an irreducible (cuspidal) representation or a sum of characters. It turns out that the former holds, and in fact this variety X contains all the cuspidal representations of $GL_2(\mathbb{F}_q)$ in its middle cohomology.

In fact, we have the following bijection between a pair $(\chi, \overline{\chi})$ and $\rho_M|H^1_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell})[\chi]$, which turns out to be the cuspidal representation of $GL_2(\mathbb{F}_q)$ typically called $\pi(\chi)$. It turns out that $\pi(\chi) \cong \pi(\overline{\chi})$, so this makes sense. Before we prove this, we quickly review the fundamental facts about the representation theory of $GL_2(\mathbb{F}_q)$.

4 Representation theory of $GL_2(\mathbb{F}_q)$

Before we discuss the representation theory of $GL_2(\mathbb{F}_q)$ it is useful to discuss its conjugacy classes. Given a matrix $M \in GL_2(\mathbb{F}_q)$, we can mostly determine its conjugacy class by its eigenvalues. If M has distinct eigenvalues in $\mathbb{F}_q$, then it is conjugate to:

$$M_{r,s} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

If M has eigenvalues $\alpha, \overline{\alpha}$ in $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$, then they are of the form $\alpha = r + s\sqrt{D}$, $\overline{\alpha} = r - s\sqrt{D}$, with D a quadratic nonresidue. Then M is conjugate to:

$$M_\alpha = \begin{pmatrix} r & Ds \\ s & r \end{pmatrix}$$

Finally, if M has repeated eigenvalue t, then M is conjugate to one of the following:

$$M_t = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$
$$M'_t = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$$

We will need these later when determining the character of a representation.

Let $T, B, P \subset GL_2(\mathbb{F}_q) = G$ be the standard torus, the standard unipotent subgroup, and the Borel subgroup respectively. It is well known that P is a normal subgroup of B and $B/P \cong T$. A character of T consists of two characters ψ, ψ' of $\mathbb{F}_q^\times$. We can extend a character (ψ, ψ') of T to B trivially, by forcing it to act trivially on P. Then we define the following principal series representation of G as $I(\psi, \psi') = Ind_B^G(\psi, \psi')$. The dimension of this representation is just the index of B in G, which is just $q + 1$.

We state the following results about $I(\psi, \psi')$ without proof.

Proposition 4.1. *The representation $I(\psi, \psi')$ is irreducible iff $\psi \neq \psi'$, otherwise it splits into a q-dimensional irreducible representation and 1-dimensional representation denoted $\overline{I}(\psi, \psi')$ and $I'(\psi, \psi')$ respectively. These two subquotients of the reducible principal series representations are called special.*

With the above proposition, we've almost enumerated all the irreducible representations of $GL_2(\mathbb{F}_q)$. It turns out that there is another class of irreducible representations called cuspidal representations which have dimension $q-1$ and whose traces are given in terms of generic characters χ of $\mathbb{F}_{q^2}^\times$. Explicitly, given a generic χ, there is an irreducible representation $\pi(\chi)$ with the following traces:

$$\begin{aligned}
Tr(\pi(\chi)(M_t)) &= (q-1)\chi(t) \\
Tr(\pi(\chi)(M_t')) &= -\chi(t) \\
Tr(\pi(\chi)(M_{r,s})) &= 0 \\
Tr(\pi(\chi)(M_\alpha)) &= -\chi(\alpha) - \chi(\overline{\alpha})
\end{aligned}$$

It is important to notice that $\pi(\chi)$ and $\pi(\overline{\chi})$ have the same traces, and thus are isomorphic as representations. In the next section, we finish our work with this variety X by proving that it contains all of these cuspidal representations in its cohomology and briefly remark on how this makes X a finite field analogue of a Shimura variety.

5 Cuspidal Representations of $GL_2(\mathbb{F}_q)$

Recall from previous sections we have ρ_C and ρ_M representations of $\mathbb{F}_{q^2}^\times$ and $GL_2(\mathbb{F}_q)$ on $H^1_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell})$. For a generic character χ of $\mathbb{F}_{q^2}^\times$, we know that ρ_M restricted to $H^1_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell})[\chi]$ is a $(q-1)$-dimensional representation of $GL_2(\mathbb{F}_q)$. With the notation from the previous section in place, we can make the following proposition. For ease of notation, set:

$$\rho = \rho_M| \left(H^1_{\acute{e}t}(X, \overline{\mathbb{Q}_\ell})[\chi]\right)$$

Proposition 5.1. *$\rho \cong \pi(\chi)$ as representations of $GL_2(\mathbb{F}_q)$.*

Proof. We know that ρ splits into irreducible representations, and since those can only have dimension $q+1, q, q-1$, or 1, it follows that if ρ isn't cuspidal, then it must be a sum of $q-1$ characters. It is easy to show that ρ is cuspidal, but we have to work out most of the traces to show that ρ is actually $\pi(\chi)$.

To show how quickly it follows that ρ is not cuspidal, we compute it trace on M_1'.

$$\begin{aligned}
Tr(\rho(M_1')) &= \frac{1}{q^2-1} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \chi(\alpha) Tr(M_1' \circ \alpha^{-1}) \\
&= \frac{1}{q^2-1} \sum_{Nm(\alpha)=1} \chi(\alpha)(2q+2-Fix(M_1' \circ \alpha^{-1}))
\end{aligned}$$

Now consider $M_1' \circ \alpha^{-1}$. If this fixed an affine point $[x, y, z]$ with $\alpha \in \mathbb{F}_q$, then we would have $y = \lambda x$, with $\lambda \in \mathbb{F}_q$. However, lines of rational slope don't intersect X, as:

$$\begin{aligned} xy^q - y^q x &= \gamma z^{q+1} \\ x^{q+1}\lambda^q - x^{q+1}\lambda &= z^q + 1 \\ 0 &= z^{q+1} \end{aligned}$$

Thus, we would have $x = y = z = 0$, which can't happen. On the other hand α has to be in $\mathbb{F}_q$, since fixing an affine point $[x, y, z]$ means $[x, y]$ is an eigenvector for M_1' with eigenvalue α^{-1}, but M_1' has eigenvalues in $\mathbb{F}_q$.

Thus, the only fixed points of this action are at infinity, where $\mathbb{F}_{q^2}^\times$ acts trivially. In order for a point at infinty to be fixed, we must have:

$$[a, b, 0] = [a + b, b, 0]$$

It is clear that only $[1, 0, 0]$ is fixed. Now we expand about the fixed point in the connected component corresponding to $\gamma \in \mathbb{F}_q^\times$. Let $Y = \frac{y}{x}$ and $Z = \frac{z}{x}$. Then formally we have the following:

$$\begin{aligned} Y^q - Y &= \gamma Z^{q+1} \\ Y &= Y^q - \gamma Z^{q+1} \\ &= (Y^q - \gamma Z^{q+1})^q - \gamma Z^{q+1} \\ &= -\gamma Z^{q+1} - \gamma Z^{q(q+1)} - \dots \end{aligned}$$

The variable Z is affected by the transformation $M_1' \circ \alpha^{-1}$ in the following way.

$$\begin{aligned} (M_1' \circ \alpha^{-1})(Z) &= \frac{z}{\alpha^{-1}(x + y)} \\ &= \alpha \frac{Z}{1 + Y} \\ &= \alpha Z - \alpha\beta Z^{q+2} - \dots \end{aligned}$$

From this we see that the index of the fixed point $[1, 0, 0]$ is 1 unless $\alpha = 1$, in which case it is $q + 2$. Now we finish the computation:

$$\begin{aligned} Tr(\rho(M_1')) &= \frac{1}{q^2 - 1}(\chi(1)(2q - 2 - (q + 2)(q - 1)) + \sum_{Nm(\alpha)=1, \alpha \neq 1} \chi(\alpha)(2q - 2 - (q - 1))) \\ &= \frac{1}{q^2 - 1}(-q^2 + q + -\chi(1)(q - 1)) \\ &= -1 \end{aligned}$$

Now suppose ρ is a sum of characters. It is well known that characters of $GL_2(\mathbb{F}_q)$ factor through the determinant map to $\mathbb{F}_q$. Using this:

$$\begin{aligned} Tr(\rho(M_1')) &= \sum_i \chi_i(M_1') \\ &= \sum_i \chi_i(det(M_1')) \\ &= \sum_i \chi_i(1) \\ &= q-1 \\ &\neq -\chi(1) \end{aligned}$$

Thus ρ is not a sum of characters, and thus must be cuspidal. To pin down which pair $(\chi, \overline{\chi})$ is associated to ρ, we compute the remaining traces.

First, we compute $Tr(\rho(M_t))$.

$$\begin{aligned} Tr(\rho(M_t)) &= \frac{1}{q^2-1} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \chi(\alpha) Tr(\alpha^{-1} \circ M_t) \\ &= \frac{1}{q^2-1} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \chi(\alpha) Tr(t\alpha^{-1}) \\ &= \frac{1}{q^2-1} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \chi(t\alpha) Tr(\alpha^{-1}) \\ &= \chi(t) \left(\frac{1}{q^2-1} \sum_{\alpha \in \mathbb{F}_{q^2}^\times} \chi(\alpha) Tr(\alpha^{-1}) \right) \\ &= \chi(t) Tr(\rho(I)) \\ &= (q-1)\chi(t) \end{aligned}$$

Now that we know what the central character is, we can compute $Tr(\rho(M_s')$ using the following facts. First observe:

$$\begin{aligned} M_s' &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{s} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \end{aligned}$$

Now we can easily compute $Tr(\rho(M_s')$ since M_s is central in $GL_2(\mathbb{F}_q)$ and therefore acts by a scalar (in this case $\chi(s)$). We have:

$$\begin{aligned} Tr(\rho(M_s')) &= Tr(\rho(M_s)\rho(M_1')) \\ &= \chi(s)Tr(\rho(M_1')) \\ &= -\chi(s) \end{aligned}$$

Finally, we compute $Tr(\rho(M_\beta))$ as follows. First we examine the fixed points of $M_\beta \circ \alpha^{-1}$. This can fix affine points whenever $\alpha = \beta, \overline{\beta}$ since we have lines of eigenvectors $y = \Lambda x$ with $\Lambda \in \mathbb{F}_{q^2}^\times$ and not $\mathbb{F}_q^\times$. Then we solve the following:

$$\begin{aligned} xy^q - y^q x &= \beta z^{q+1} \\ (\Lambda^q - \Lambda)x^{q+1} &= \gamma z^{q+1} \end{aligned}$$

This clearly has $q+1$ solutions in projective coordinates, each giving a fixed point of index 1 in each connected component. Now we consider fixed points at infinity. Suppose $M_\beta = \begin{pmatrix} a & Db \\ b & a \end{pmatrix}$. Then we want to know if $[x, y, 0]$ can be fixed by this matrix. If $y = 0$, then $[x, 0, 0] \neq [ax, bx, 0]$ since $b \neq 0$. If $y \neq 0$, then $[x, y, 0] = [ax+Dby, bx+ay, 0]$ iff:

$$bx^2 + axy = axy + Dby^2$$

However this is impssible since D isn't a square. Thus, no points at infinity are fixed. Now we can finish the computation as follows:

$$\begin{aligned} Tr(\rho(M_\beta)) &= \frac{1}{q^2-1} \sum_{Nm(\alpha)=Nm(\beta)} \chi(\alpha)(2q - 2 - Fix(M_\beta \circ \alpha^{-1})) \\ &= \frac{1}{q^2-1}((\chi(\beta) + \chi(\overline{\beta}))(2q - 2 - (q+1)(q-1)) - (\chi(\beta) + \chi(\overline{\beta}))(2q-2) \\ &= -(\chi(\beta)) + \chi(\overline{\beta}) \end{aligned}$$

Now we can finally show $\rho \cong \pi(\chi)$. Since we know ρ is cuspidal, $Tr(\rho(M_{a,b})) = 0$. Thus, since the traces of ρ and $\pi(\chi)$ agree on conjugacy class representatives of $GL_2(\mathbb{F}_q)$, they must be isomorphic. □

Before moving on, we have a final remark.

Remark 5.2. *This variety X is a finite field analogue of a Shimura variety in the sense that we have exhibited a Langlands correspondence in its cohomology. The correspondence comes from the following idea. Let $\mathbb{Q}_{p^2}$ be the unique unramified quadratic extension of $\mathbb{Q}_p$. One way of producing a 2-dimensional representation of $G\mathbb{Q}_p$ is to induce a character of $G\mathbb{Q}_{p^2}$. A character of $G\mathbb{Q}_{p^2}$ factors through its abelianization, which by local class field theory is canonically isomorphic to $\mathbb{Q}_{p^2} \cong \mathbb{Z} \times \mathbb{Z}_{p^2}$. If we force the character to be continuous, then it factors through a root of unity in the $\mathbb{Z}$ component* *and through a finite quotient of $\mathbb{Z}_{p^2}$, of which $\mathbb{F}_{q^2}^{\times}$ is an example. So we've exhibited a correspondence between (certain) 2-dimensional representations of $W\mathbb{Q}_p$ and (automorphic, cuspidal) representations of $GL_2(\mathbb{F}_q)$, and we've constructed this correspondence in the cohomology of our variety X.*

6 Gauss Sums and Cohomology

Consider varieties of the following form for $Q(\vec{x})$ a nondegenerate quadratic form over $\mathbb{F}_q$:

$$Y(Q) : y^q - y = Q(\vec{x})$$

We wish to determine for which Q and $k \geq 1$ is $Y(Q)$ maximal over $\mathbb{F}_{q^k}$. The variety $Y(Q)$ has a few group actions on it. It has an action of $\mathbb{F}_q$ by $a \cdot (y, \vec{x}) = (y + a, \vec{x})$. It also has an action of $O(Q)$, the group of linear transformations preserving the quadratic form Q. Finally, as all varieties over finite fields do, it has an action of $Frob_q^{\mathbb{Z}}$. Since $\mathbb{F}_q$ acts on $Y(Q)$ by $\mathbb{F}_q$-linear automorphisms, its action commutes with $Frob_q^{\mathbb{Z}}$, which will allow us to use similar ideas from the previous section to better understand the trace of $Frob_q$ on the cohomology of $Y(Q)$, and thus determine its maximality. A result of Nick Katz from his paper "Crystalline Cohomology" in line with the above statements is useful to state here.

Theorem 6.1. *Let $Y/\mathbb{F}_q$ be projective and smooth, and G a finite group acting on Y by $\mathbb{F}_q$-linear automorphisms, and ρ an irreducible complex (or ℓ-adic) representation of G. Define:*

$$S(Y/\mathbb{F}_q, \rho, n) = \frac{1}{\#G} \sum_{g \in G} Tr(\rho(g)) \# Fix(Frob_q^n \circ g^{-1})$$

Then the following are equivalent:

(1) The multiplicity of ρ is one in $H^{i_0}_{\acute{e}t}(Y(Q), \overline{\mathbb{Q}_\ell})$ and zero in $H^{i}_{\acute{e}t}(Y(Q), \overline{\mathbb{Q}_\ell})$ for $i \neq i_0$.

(2) For all $n \geq 1$, we have:

$$|S(Y/F_q, \rho, n)| = (\sqrt{q})^{i_0 n}$$

(3) $Frob_q$ acts on $H^{i}_{\acute{e}t}(Y(Q), \overline{\mathbb{Q}_\ell})$ by the scalar $(-1)^{i_0} S(Y/\mathbb{F}_q, \rho, 1)$.

In the case of $Y = Y(Q)$, $G = \mathbb{F}_q$, and $\rho = \psi$ we will see that $S(Y(Q)/\mathbb{F}_q, \chi, n)$ is a power of a Gauss sum, which we will compute and use to determine when $Y(Q)$ is maximal. To ease our computations, we use the following result about quadratic forms over $\mathbb{F}_q$ from Serre's "A Course in Arithmetic" without proof.

Proposition 6.2. *Every nondegenerate quadratic form over $\mathbb{F}_q$ is equivalent to $X_1^2 + \dots cX_n^2$ where $c \in \mathbb{F}_q^\times$. This gives two equivalence classes, depending on whether or not c is a quadratic residue. When we want to emphasize it, we will write $Y(Q_R)$ and $Y(Q_N)$ for the quadratic residue and nonresidue classes respectively.*

With these propositions in place, we compute $S(Y(Q)/\mathbb{F}_q, \chi, n)$. The key to realizing this quantity as being related to Gauss sums is by computing the quantity $Fix(Frob_q^n \circ [-a])$. A point $(y, \vec{x})$ is fixed when $(y, \vec{x}) = (y^{q^n} - a, \vec{x}^{q^n})$. This means each component of $\vec{x}$ is in $\mathbb{F}_{q^n}$ and that $y^{q^n} - y = a$. Now, we recognize that the left hand side is the trace of $y^q - y$ from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$. Thus, whenever $Tr(Q(\vec{x})) = a$, we get q fixed points, corresponding to the q distinct solutions to $y^q - y = a$. Define:

$$I(\vec{x}, a) = \begin{cases} q & \text{if } Tr(Q(\vec{x})) = a \\ 0 & \text{otherwise} \end{cases}$$

Then we have the following:

$$\begin{aligned} S(Y(Q)/\mathbb{F}_q, \psi, n) &= \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi(a) \# Fix(Frob_q^n \circ [-a]) \\ &= \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi(a) \sum_{\vec{x} \in (\mathbb{F}_q^n)^d} I(\vec{x}, a) \\ &= \frac{1}{q} \sum_{\vec{x} \in (\mathbb{F}_q^n)^d} \sum_{a \in \mathbb{F}_q} \psi(a) I(\vec{x}, a) \\ &= \sum_{\vec{x} \in (\mathbb{F}_q^n)^d} \psi(Tr(Q(\vec{x}))) \\ &= \left(\prod_{a=1}^{d-1} \sum_{x \in \mathbb{F}_{q^n}} \psi(Tr(x^2)) \right) \sum_{x \in \mathbb{F}_{q^n}} \psi(cTr(x^2)) \end{aligned}$$

By changing the order of summation and applying Fubini's theorem, we have turned this sum into a product of sums over finite fields, which will turn out to be Gauss sums. Let $\left(\frac{x}{q}\right)$ be the $\mathbb{F}_q$-Legendre symbol. Then using the structure of $\mathbb{F}_q^\times$ its clear that $\left(\frac{x}{q^k}\right) = \left(\frac{Nm(x)}{q}\right)$, where Nm denotes the norm from $\mathbb{F}_{q^k}$ to $\mathbb{F}_q$. Then we can manipulate these sums to show that they are in fact Gauss sums in the character sums sense:

$$\begin{aligned} \sum_{x \in \mathbb{F}_{q^n}} \psi(Tr(x^2)) &= \sum_{x \in \mathbb{F}_{q^n}} \left(\frac{x}{q^n}\right) \psi(Tr(x)) \\ &= \sum_{x \in \mathbb{F}_{q^n}} \left(\frac{Nm(x)}{q}\right) \psi(Tr(x)) \\ &:= g_\psi(n) \end{aligned}$$

We are now in the lucky situation of having a Gauss sum on $\mathbb{F}_{q^n}$ where the multplicative and additive characters factor through the respective norm and trace maps to $\mathbb{F}_q$. The Hasse-Davenport relation tells us the following:

$$-g_\psi(n) = (-1)^n g_\psi(1)$$

Interestingly, we can actually give a "geometric" proof of the Hasse-Davenport relation using the variety $Y : y^q - y = x^2$. For a nontrivial character ψ on $\mathbb{F}_q$, consider:

$$\begin{aligned} S(Y, \psi, 1) &= \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi(a) \# fix(Frob_q \circ [-a]) \\ &= \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi(a) q \left(1 + \left(\frac{a}{q}\right)\right) \\ &= \sum_{a \in \mathbb{F}_q} \psi(a) \left(\frac{a}{q}\right) \\ &:= g_\psi \end{aligned}$$

We know that $H^1_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})[\psi]$ is one-dimensional, and the trace of $Frob_q$ on $H^1_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})[\psi]$ is $(-1)^1 g_\psi$. Thus, the trace of $Frob_q^n$ on $H^1_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})[\psi]$ is just $(-1)^n g_\psi^n$, since $H^1_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})[\psi]$ is one-dimensional. On the other hand, the trace of $Frob_q^n$ on $H^1_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})[\psi]$ is just $(-1)^1 S(Y/\mathbb{F}_{q^n}, \psi, 1) = S(Y/\mathbb{F}_q, \psi, n) = -g_\psi(n)$. Equating these two gives the Hasse-Davenport relation.

In order to determine the maximality of $Y(Q)$, we need to be able to evaluate $S(Y(Q), \psi, 1)$, which boils down to computing certain Gauss sums over $\mathbb{F}_q$. It is well-known that the characters ψ of $\mathbb{F}_q$ are of them form $\psi_b(a) = \zeta_p^{Tr(ab)}$ for some $b \in \mathbb{F}_q$. Thankfully, there are only two different flavors of Gauss sums, in the following sense. Write $q = p^r$, then:

Proposition 6.3. *For ψ_b a non-trivial character of $\mathbb{F}_q$ and b a quadratic residue, its associated Gauss sum has the following value:*

$$g_{\psi_b} = (-1)^{r+1} (\sqrt{p^*})^r$$

For c a quadratic nonresidue:

$$g_{\psi_c} = -g_{\psi_b}$$

For the trivial character, ψ_0, we have:

$$g_{\psi_0} = q$$

Proof. To do this, we compute recall the computation of the standard Gauss sums over $\mathbb{F}_p$ and appeal to the Hasse-Davenport relation. For this section, we use η to denote a character on $\mathbb{F}_p$. Recall the following equality:

$$\sum_{a\in\mathbb{F}_p}\left(\frac{a}{p}\right)\zeta_p^a = \sqrt{p^*}$$

Proving this equality up to sign is elementary and well-known, whereas getting the sign correct is more difficult and is carried out in various texts. Now suppose b is a quadratic residue and consider the Gauss sum associated to η_b:

$$\begin{aligned}
g_{\eta_b} &= \sum_{a\in\mathbb{F}_p}\left(\frac{a}{p}\right)\zeta_p^{ab} \\
&= \sum_{a\in\mathbb{F}_p}\left(\frac{ab^{-1}}{p}\right)\zeta_p^{a} \\
&= \left(\frac{b^{-1}}{p}\right)\sum_{a\in\mathbb{F}_p}\left(\frac{a}{p}\right)\zeta_p^{a} \\
&= g_{\eta_1} \\
&= \sqrt{p^*}
\end{aligned}$$

Now consider the sum g_{ψ_r} for m a quadratic residue. As above, we can show:

$$g_{\psi_m} = g_{\psi_1}$$

Now we consider g_{ψ_1}:

$$\begin{aligned}
g_{\psi_1} &= \sum_{x\in\mathbb{F}_q}\zeta_p^{Tr(1\cdot x)}\left(\frac{x}{q}\right) \\
&= (-1)^{r+1}g_{\eta_1} \\
&= (-1)^{r+1}\sqrt{p^*}
\end{aligned}$$

As before, for n a quadratic nonresidue, $g_{\psi_n} = -g_{\psi_1}$. Finally, the statement $g_{\psi_0} = q$ is clear, as each term in the sum is 1 and there are q of them.

□

Now we can determine under what conditions $Y(Q)$ is maximal over $\mathbb{F}_q^k$. As before, write $q = p^r$, and let consider $Y(Q) : y^q - y = x_1^2 + \dots cx_d^2$.

Proposition 6.4. *1. When $p \equiv 1 \pmod 4$ The variety $Y(Q_R)$ is maximal over $\mathbb{F}_q^k$ iff d is even.*

2. When $p \equiv 3 \pmod 4$*, the variety* $Y(Q_R)$ *is maximal over* $\mathbb{F}_q^k$ *iff* $k \equiv 2 \pmod 4$ *and* r, d *are odd, or* d *is even and* $rdk \equiv 0 \pmod 4$*.*

3. When $p \equiv 1 \pmod 4$*, the variety* $Y(Q_N)$ *is maximal over* $\mathbb{F}_q^k$ *iff* k *and* d *are even.*

4. When $p \equiv 3 \pmod 4$*, the variety* $Y(Q_N)$ *is maximal over* $\mathbb{F}_q^k$ *iff exactly one of* k *or* d *is* $2 \pmod 4$ *and the others are odd, or* d *and* k *are even.*

Proof. In order for $Y(Q)$ to be maximal over $\mathbb{F}_q^k$, we need that for every i, $Frob_q^k$ acts on $H^i_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})$ by the scalar $(-1)^i q^{\frac{ik}{2}}$. For every nontrivial character ψ, we will show that $|S(Y, \psi, n)| = (\sqrt{q})^{nd}$, which tells us by Katz's result that ψ only appears in $H^d_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})$, and with multiplicity 1. For the trivial character, ψ_0, we will show that $|S(Y, \psi_0, n)| = (\sqrt{q})^{2nd}$, so that ψ_0 only appears in $H^{2d}_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})$ and also with multiplicity 1. Then we will need to show that $Frob_q^k$ acts as the correct scalar precisely when the conditions of the theorem hold. Let ψ_a be any nontrivial character.

$$
\begin{aligned}
|S(Y(Q), \psi_a, n)| &= |g_{\psi_a}(n)^{d-1} \cdot g_{\psi_{ac}}(n)| \\
&= (\sqrt{q})^{n(d-1)} \cdot (\sqrt{q})^n \\
&= (\sqrt{q})^{nd}
\end{aligned}
$$

Similarly, if $a = 0$, each g_{ψ_0} is an n^{th} power of the trivial Gauss sum, which is q. That gives:

$$
\begin{aligned}
|S(Y(Q), \psi_0, n)| &= q^{nd} \\
&= (\sqrt{q})^{2nd}
\end{aligned}
$$

Now we know that $Y(Q)$ only has cohomology in degrees d and $2d$, so to determine its maximality, we only need to show that $Frob_q$ acts on $H^d_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})$ and $H^{2d}_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})$ via the appropriate scalars. We know from Katz that $Frob_q$ acts on the one-dimensional space $H^d_{ét}(Y(Q), \overline{\mathbb{Q}_\ell})[\psi]$ via the scalar $(-1)^d S(Y(Q), \psi, 1)$. Thus we need to determine conditions for so that for all nontrivial ψ:

$$
((-1)^d S(Y(Q), \psi, 1))^k = (-1)^d (\sqrt{q})^d
$$

Also we need that for the trivial character ψ_0:

$$
S(Y(Q), \psi_0, 1)^k = (\sqrt{q})^{2dk}
$$

The first is easier, and is as follows:

$$
\begin{aligned}
S(Y(Q), \psi_0, 1)^k &= (g_{\psi_0}^{d-1} \cdot g_{\psi_{0 \cdot c}})^k \\
&= g_{\psi_0}^{kd} \\
&= q^{kd}
\end{aligned}
$$

Thus we see that $Frob_q$ acts by the correct scalar without any condition on r or d. Now we do the other computation first for $Y(Q_R)$. Let $m, n \in \mathbb{F}_q^\times$ be a quadratic residue and nonresidue respectively.

$$((-1)^d S(Y(Q_R), \psi_m, 1))^k = (-1)^{rdk}(\sqrt{p^*})^{rdk} \tag{5}$$
$$((-1)^d S(Y(Q_R), \psi_n, 1))^k = (-1)^{rdk+dk}(\sqrt{p^*})^{rdk} \tag{6}$$

In particular, these two quantities must be equal, so:

$$dk \equiv 0 \pmod 2$$

Recall that we need these quantities to be equal to $(-1)^d(\sqrt{p})^{rdk}$. Thus, when $p \equiv 1 \pmod 4$, we see that it is enough for d to be even for sums (5) and (6) to be equal to $(-1)^d(\sqrt{p})^{rdk}$. For $p \equiv 3 \pmod 4$, if d is even, then we only need $rdk \equiv 0 \pmod 4$, and if d is odd, then we must have k even since dk is even. However, we also need $rdk \equiv 2 \pmod 4$, which means $k \equiv 2 \pmod 4$ and r is also odd. This is what we sought to prove.

Now we consider $Y(Q_N)$, and thus have to analyze the following quantities:

$$\begin{aligned} ((-1)^d S(Y(Q_N), \psi_m, 1))^k &= (-1)^d(g_{\psi_m}^{d-1} g_{\psi_n})^k \\ &= (-1)^{rdk+k}(\sqrt{p^*})^{rdk} \\ ((-1)^d S(Y(Q_N), \psi_n, 1))^k &= ((-1)^d g_{\psi_n}^{d-1} \cdot g_{\psi_m})^k \\ &= (-1)^{dk+rdk+k}(\sqrt{p^*})^{rdk} \end{aligned}$$

Again, comparing the terms we see that $dk \equiv 0 \pmod 2$. For $Y(Q_N)$ to be maximal, we need:

$$(-1)^k(\sqrt{p^*})^{rdk} = (-1)^d(\sqrt{p})^{rdk}$$

If $p \equiv 1 \pmod 4$, then we need d and k to have the same parity, and thus they must both be even, since dk is even. This is the only condition. If $p \equiv 3 \pmod 4$, then first consider $rdk \equiv 2 \pmod 4$. In this case, we would need k and d to have opposite parity, which means exactly one of them is 2 (mod 4), and the others are odd. If $rdk \equiv 0 \pmod 4$, then k and d have to have the same parity, and thus are both even, and this is the only condition. This is what we wanted to prove. □

7 Future Directions and Acknowledgements

There are several future directions that can be pursued in this area. For example, what other interesting varieties like X with commuting group actions can be written down and easily computed with, and can we hope to obtain similarly interesting results about the representation theory of these groups and maximality of these varieties. For example, the varieties $Y(Q)$ have the action of an orthogonal group which could have interesting representation theoretic consequences.

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